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# ELECTROELASTIC WAVES IN POLARIZING MEDIA*

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A system of equations is considered that describes a certain class of polarizing media in an electromagnetic field with both the spatial inhomogeneities and the relaxation processes of electrical polarization of the media taken into account. The nonlinear and the linearized theories are elucidated within the framework of the electrostatic approximation. Solutions are given for the equations considered in an electrostatic approximation, in the form of electroelastic waves. Without taking the relaxation of the electrical polarization into account, electorelastic waves were considered in piezoceramic media in /1/ and in ferroelectrics in /2/.

**1. Models of polarizing media in an electromagnetic field.** Let $\partial_\alpha$, $x^\alpha$ $(\alpha = 1, 2, 3)$ be the bases and variables of the Cartesian coordinate system of the observer in the three-dimensional, physical (Euclidean) space $V$, and $\partial_\alpha^\wedge$, $\xi^\alpha$ are the bases and variables of the coordinate system of the accompanying (Lagrange) space for the continuous medium considered in $V$. Let us define the mass density of the medium $\rho$, the finite strain tensor $\varepsilon^{\wedge\alpha\beta}\partial_\alpha^\wedge\partial_\beta^\wedge$ and the velocity vector of the medium $v = v^\alpha\partial_\alpha$ by the relationships

$$\rho = \rho_0 (g^\wedge)^{-1/2}, \quad \varepsilon_{\alpha\beta}^\wedge = \frac{1}{2}(g_{\alpha\beta}^\wedge - \overset{\circ}{g}_{\alpha\beta}), \quad v^\alpha = \frac{dx^\alpha}{dt}, \quad g^\wedge = \det \| g_{\alpha\beta}^\wedge \| \tag{1.1}$$

Here $g_{\alpha\beta}^\wedge$ are the metric tensor components in the accompanying coordinate system, $g_{\alpha\beta}^\circ$ are the metric tensor components of the space of initial states defined on the manifold $\xi^\alpha$, and $d/dt$ is the symbol of the substantive derivative with respect to time $t$ (for constant Lagrange variables $\xi^\alpha$).

Let $P = P^\alpha\partial_\alpha$ be the three-dimensional electrical polarization vector of the medium which is invariant relative to the selection of the inertial coordinate system of the observer /3,4/, $E = E^\alpha\partial_\alpha$ is the electrical field intensity vector, $D = D^\alpha\partial_\alpha$ the electrical inducation vector, $H = H^\alpha\partial_\alpha$ the magnetic field intensity vector, and $B = B^\alpha\partial_\alpha$ the magnetic induction vector. Polarizing media for which the magnetization is zero by the condition in the intrinsic basis, will be considered below. In this case the vectors $D$ and $H$ are related to the vectors $E$, $B$, $P$ by the equation ($c$ is the speed of light in vacuum)

$$D = E + 4\pi P, \quad H = B + 4\pi/c\ [v, P] \tag{1.2}$$

The vectors $E$ and $B$ can be expressed in terms of the scalar and vector potentials by the equalities

$$B = \operatorname{rot} A, \quad E = -\operatorname{grad}\varphi - \frac{1}{c}\frac{\partial}{\partial t}A \tag{1.3}$$

Here $\partial/\partial t$ is the symbol of partial differentiation with respect to time for constant variables $x^\alpha$.

Let us consider the class of models of a polarizing medium in an electromagnetic fields described by the system of dynamic equations:

$$\operatorname{div} D = 0, \quad \operatorname{rot} H = \frac{1}{c}\frac{\partial}{\partial t}D, \quad \operatorname{rot} E = -\frac{1}{c}\frac{\partial}{\partial t}B, \quad \operatorname{div} B = 0 \tag{1.4}$$

$$\frac{\partial}{\partial t}\rho v_\alpha = \partial_\beta P^\beta_{\alpha(m)} + Q_\alpha + \frac{1}{8\pi}(B_\lambda\partial_\alpha H^\lambda - H^\lambda\partial_\alpha B_\lambda + D_\lambda\partial_\alpha E^\lambda - E^\lambda\partial_\alpha D_\lambda)$$

$$E^\alpha + \frac{1}{c}[v, B]^\alpha + \frac{\partial\Lambda_0}{\partial P_\alpha} - \partial_\lambda\frac{\partial\Lambda_0}{\partial\partial_\lambda P_\alpha} = \Pi^\alpha$$

$$\rho T\frac{ds}{dt} = -\partial_\alpha q^\alpha + \tau^{*\alpha\beta}e_{\alpha\beta} + \Pi^\alpha\left(\frac{d}{\partial t}P_\alpha - [\omega, P]_\alpha\right)$$

$$\rho T + \frac{\partial\Lambda_0}{\partial s} = 0, \quad \frac{d\rho}{dt} + \rho\operatorname{div} v = 0$$

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The symbol $\partial_\alpha = \partial/\partial x^\alpha$ denotes the partial derivative with respect to the variable $x^\alpha$, $\omega = {}^1/_2 \operatorname{rot} v$ is the vortex velocity vector, $e_{\alpha\beta} = {}^1/_2(\partial_\alpha v_\beta + \partial_\beta v_\alpha)$ are the strain rate tensor components, $Q_\alpha$ are components of the external volume force vector acting on the medium, $T$ is the temperature in the equilibrium processes, $s$ is the specific entropy density, $\tau^{*\alpha\beta}$ are components of the viscous stress tensor, $q^\alpha$ are components of the thermal flux vector, and the components of the vector $\Pi^\alpha$ govern the relaxation process of the electrical polarization of the medium. The components of the tensor $P^\beta_{\alpha(m)}$ in equations (1.4) are defined by the relationship

$$P^\beta_{\alpha(m)} = -\rho v_\alpha v^\beta - \frac{\partial \Lambda_0}{\partial x_\lambda{}^\alpha} x_\lambda{}^\beta + \frac{\partial \Lambda_0}{\partial \partial_\beta P_\lambda} \partial_\alpha P_\lambda - \delta_\alpha{}^\beta \left[\Lambda_0 + \frac{1}{2} P_\lambda \left(E^\lambda + \frac{1}{c}[v, B]^\lambda\right)\right] + \tau^{*\beta}_\alpha + \frac{1}{2}(P_\alpha \Pi^\beta - P^\beta \Pi_\alpha) \quad (1.5)$$

where $x_\gamma{}^\alpha = \partial x^\alpha / \partial \xi^\gamma$ is the distortion, $\Lambda_0$ is a given function of the system of arguments

$$x_\gamma{}^\alpha,\ P_\alpha,\ \partial_\beta P_\alpha,\ s,\ K_C \quad (1.6)$$

and $K_C$ are given constant tensors $(dK_C/dt = 0)$ governing the anisotropy of the medium, for instance.

Equations (1.4) and (1.5) can be obtained from the variational equation /3–9/. In this case the function $\Lambda_0$ is a part of the Lagrangian. The equations being considered here differ from /3/ only in that the more general Onsager relationships governing the relaxation term $\Pi^\alpha$ are used later.

The system of equations (1.4) and (1.5) contains the Maxwell equations for the electromagnetic field in a medium, the momentum equations, the equation for electrical polarization of the medium, the continuity equation for the mass density of the medium, the entropy balance equation, and the equation for the temperature. The energy equation

$$\frac{\partial}{\partial t}\left\{\varepsilon_{(m)} + \frac{1}{8\pi}(D_\alpha E^\alpha + H_\alpha B^\alpha) + \frac{1}{c} v_\alpha [B, P]^\alpha\right\} + \partial_\beta \left\{\varepsilon^\beta_{(m)} + \frac{c}{4\pi}[E, H]^\beta + \frac{1}{c} v^\beta v_\alpha [B, P]^\alpha\right\} = Q_\alpha v^\alpha \quad (1.7)$$

in which the volume energy density of the medium $\varepsilon_{(m)}$ and the energy flux vector components of the medium $\varepsilon^\beta_{(m)}$ are defined by the relationships

$$\varepsilon_{(m)} = \frac{1}{2}\rho v^2 - \frac{1}{2} P_\alpha \left(E^\alpha + \frac{1}{c}[v, B]^\alpha\right) - \Lambda_0 \quad (1.8)$$

$$\varepsilon^\beta_{(m)} = q^\beta + \varepsilon_{(m)} v^\beta - (\rho v_\alpha v^\beta + P^\beta_{\alpha(m)}) v^\alpha + \frac{\partial \Lambda_0}{\partial \partial_\beta P_\alpha} \frac{d}{dt} P_\alpha$$

follows from (1.4) and (1.5).

Components of the viscous stress tensor $\tau^{*\alpha\beta}$, components of the thermal flux vector $q^\alpha$, and the relaxation term $\Pi^\alpha$ of the equation describing the polarization of the medium should be given to close the system of equations (1.4) and (1.5). On the basis of the expression for the internal entropy production $d_i s/dt$, which can be given in the Onsager form

$$\rho T \frac{d_i S}{dt} = -\frac{1}{T} q^\alpha \partial_\alpha T + \tau^{*\alpha\beta} e_{\alpha\beta} + \Pi^\alpha \left(\frac{d}{dt} P_\alpha - [\omega, P]_\alpha\right) \quad (1.9)$$

by definition, the following estimations for the quantities $\tau^{*\alpha\beta}, \Pi^\alpha, q^\alpha$ can be taken for example:

$$\tau^{*\alpha\beta} = \tau^{\alpha\beta\lambda\theta} e_{\lambda\theta} + b^{\alpha\beta\lambda}\left(\frac{d}{dt} P_\lambda - [\omega, P]_\lambda\right) \quad (1.10)$$

$$\Pi^\alpha = s^{\alpha\beta}\left(\frac{d}{dt} P_\beta - [\omega, P]_\beta\right) + s^{\alpha\beta\lambda} e_{\beta\lambda} + m^{\alpha\beta}\partial_\beta T$$

$$q^\alpha = -\varkappa^{\alpha\beta}\partial_\beta T + \eta^{\alpha\beta}\left(\frac{d}{dt} P_\beta - [\omega, P]_\beta\right)$$

The coefficients $\tau, b, s, m, \eta, \varkappa$ in (1.10) can be given in the form of functions of the governing parameters of the medium and the field in such a way that the condition $d_i s/dt \geqslant 0$ would be satisfied.

For fixed bodies the relaxation of electrical polarization is ordinarily determined by the relaxation term $\Pi^\alpha \sim dP^\alpha/dt$. The expression (1.9) postulated here for the internal entropy production because of polarization for moving media is based substantially on the generalized expression for $\Pi^\alpha$ in the form $\Pi^\alpha \sim d'P^\alpha/dt = dP^\alpha/dt - [\omega, P]^\alpha$, where $d'/dt$ is the time derivative in a basis that moves and rotates with the particle of the medium.

Physical specification of the class of models described by equations (1.4), (1.5), (1.9) and (1.10) is associated with giving definite form to the functions $\Lambda_0, q^\alpha, \Pi^\alpha, \tau^{*\alpha\beta}$. In particular, cases when the quantities $x^\alpha_\lambda$ are in the function $\Lambda_0$ in terms of the strain tensor components $\varepsilon_{\alpha\beta}^{\wedge}$ correspond to models of elastic polarizing media. Cases when the quantities $x^{\beta}{}_{\alpha}$ are in $\Lambda_0$ in terms of the mass density of the fluid $\rho$ correspond to models of liquid polarizing media. For example, if $\Lambda_0 = \Lambda_0(\rho, s, P_\alpha, K_C)$, then we have for the components of the tensor $P^{\beta}_{\alpha(m)}$ in the momentum equations in (1.4)

$$P^{\beta}_{\alpha(m)} = -\rho v_\alpha v^\beta - p\delta_\alpha{}^\beta - \tau^{*\beta}_{\alpha} + \frac{1}{2}(P_\alpha \Pi^\beta - P^\beta \Pi_\alpha) \qquad (1.11)$$

$$p = -\rho^2 \frac{\partial \Lambda_0/\rho}{\partial \rho} + \frac{1}{2} P_\lambda \left(E^\lambda + \frac{1}{c}[v, B]^\lambda\right)$$

In the electrostatic approximation the system (1.4) becomes

$$\operatorname{div} D = 0, \ \operatorname{rot} E = 0, \ \frac{d\rho}{dt} + \rho \operatorname{div} v = 0, \quad \frac{\partial}{\partial t}\rho v_\alpha = \partial_\beta P^{\beta}_{\alpha(m)} + Q_\alpha + \frac{1}{2}(P_\lambda \partial_\alpha E^\lambda - E^\lambda \partial_\alpha P_\lambda) \qquad (1.12)$$

$$E^\alpha + \frac{\partial \Lambda_0}{\partial P_\alpha} - \partial_\lambda \frac{\partial \Lambda_0}{\partial \partial_\lambda P_\alpha} = \Pi^\alpha, \quad \rho T + \frac{\partial \Lambda_0}{\partial s} = 0, \quad \rho T \frac{ds}{dt} = -\partial_\alpha q^\alpha + \tau^{*\alpha\beta} e_{\alpha\beta} + \Pi^\alpha \left(\frac{d}{dt} P_\alpha - [\omega, P]_\alpha\right)$$

The components of the tensor $P^{\beta}_{\alpha(m)}$ in (1.12) are defined by (1.5).

In the electrostatic approximation the components of the electrical intensity vector $E$ are expressed in terms of the potential $\varphi$ by the equation $E = -\operatorname{grad} \varphi$ consequently an equation for the potential $\varphi$

$$\Delta \varphi = 4\pi \operatorname{div} P \qquad (1.13)$$

can be taken in place of the Maxwell equations in (1.12).

2. Linearized theory of polarizing media in the electrostatic approximation. Let us define the displacement vector of points of the media $u = u^\alpha \partial_\alpha$ by the equality $u = r - r_0$ in which $r$ is a radius-vector of points of the medium at the current time, and $r_0$ is a radius-vector of points of the medium in the initial state. We shall furthermore consider that the Cartesian coordinate system of the observer and the accompanying coordinate system agree at the initial time $t = t_0$. Let us examine the motion of the medium for which the gradient of the displacement vector and the gradient of the polarization vector of the medium are small, while the components of the polarization vector $P$, the temperature $T$, the entropy $s$, the mass density of the medium $\rho$ and the components of the electrical induction vector $E$ vary little relative to the equilibrium (constant) values $P_0$, $T_0$, $s_0$, $\rho_0$, $E_0$. Assuming that

$$P^\alpha = P_0{}^\alpha + p^\alpha, \quad s = s_0 + s_1, \quad \rho = \rho_0 + \rho_1, \quad E^\alpha = E_0{}^\alpha + e^\alpha \qquad (2.1)$$

where $p^\alpha, s_1, \rho_1, e^\alpha$ are small quantities, considered as first order infinitesimals, the function $\Lambda_0$ can be expanded in series in $u_{\alpha\beta} = \partial_\beta u_\alpha, \partial_\alpha p_\beta, p_\beta, s_1$. Limiting ourselves to second order of smallness in such an expansion, we obtain

$$-\Lambda_0 = \frac{1}{2}\lambda^{\alpha\beta\lambda\theta} u_{\alpha\beta} u_{\lambda\theta} + \frac{1}{2}\alpha^{\alpha\beta\lambda\theta}\partial_\alpha p_\beta \partial_\lambda p_\theta + \mu^{\alpha\beta\lambda\theta} u_{\alpha\beta}\partial_\lambda p_\theta + \qquad (2.2)$$

$$\zeta^{\alpha\beta\lambda} u_{\alpha\beta} p_\lambda + \theta^{\alpha\beta\lambda} p_\alpha \partial_\beta p_\lambda + n^{\alpha\beta} u_{\alpha\beta} s_1 + \nu^{\alpha\beta} s_1 \partial_\alpha p_\beta + \frac{1}{2}\nu s_1{}^2 +$$

$$\frac{1}{2}\beta^{\alpha\beta} p_\alpha p_\beta + \beta^\alpha p_\alpha s_1 + Q^{\alpha\beta} u_{\alpha\beta} + C^{\alpha\beta}\partial_\alpha p_\beta + C^\alpha p_\alpha + \rho_0 T_0 s_1 + \text{const}$$

The constant coefficients of the small quantities in formula (2.2) can be expressed in terms of the function $\Lambda_0$ and the derivatives of $\Lambda_0$ evaluated in the initial state. If (2.2) for $\Lambda_0$ is not taken with respect to exact nonlinear theory, then the specific values of the coefficients in (2.2) can be associated with the additional assumptions, in particular, with respect to the symmetry properties of the medium. If the stresses in the medium are zero in the initial state, then $Q^{\alpha\beta} = 0$ should be inserted in the expansion (2.2). According to the definition, the coefficients $\lambda$, $\alpha$, $\beta$ in (2.2) always possess the following symmetry properties

$$\lambda^{\alpha\beta\lambda\theta} = \lambda^{\lambda\theta\alpha\beta}, \quad \alpha^{\alpha\beta\lambda\theta} = \alpha^{\lambda\theta\alpha\beta}, \quad \beta^{\alpha\beta} = \beta^{\beta\alpha}$$

Furthermore, we take

$$\tau^{*\alpha\beta} = \tau^{\alpha\beta\lambda\theta} e_{\lambda\theta}, \quad q^\alpha = -\varkappa^{\alpha\beta}\partial_\beta T, \quad \Pi^\alpha = s^{\alpha\beta}\left(\frac{d}{dt} P_\beta - [\omega, P]_\beta\right) \qquad (2.3)$$

for the quantities $\tau^{*\alpha\beta}$, $q^\alpha$, $\Pi^\alpha$ in $\delta W^*$.

In the presence of the relationships (2.3), the linearized equations corresponding to the function $\Lambda_0$ defined by (2.2) have the form

$$\operatorname{rot} e = 0, \quad \operatorname{div}(e + 4\pi p) = 0 \tag{2.4}$$

$$e^{\alpha} - \beta^{\alpha\beta} p_{\beta} - Q^{\alpha\beta\lambda}\partial_{\beta} p_{\lambda} - \zeta^{\mu\nu\alpha} u_{\mu\nu} + \alpha^{\beta\alpha\lambda\theta}\partial_{\beta}\partial_{\lambda} p_{\theta} - \beta^{\alpha} s_1 +$$
$$\mu^{\mu\nu\lambda\alpha}\partial_{\lambda} u_{\mu\nu} + Q^{\mu\beta\alpha}\partial_{\beta} p_{\mu} + \nu^{\beta\alpha}\partial_{\beta} s_1 = s^{\alpha\beta}\left(\frac{\partial}{\partial t} p_{\beta} - [\omega, P_0]_{\beta}\right)$$

$$\rho_0 \frac{\partial^2 u^{\alpha}}{\partial t^2} = Q^{\alpha} + \partial_{\beta}\Big[\lambda^{\alpha\beta\lambda\theta} u_{\lambda\theta} + \mu^{\alpha\beta\lambda\theta}\partial_{\lambda} p_{\theta} + \zeta^{\alpha\beta\lambda} p_{\lambda} +$$
$$(\eta^{\alpha\beta} + \rho_0 T_0 \delta^{\alpha\beta}) s_1 + \tau^{\alpha\beta\lambda\theta}\frac{\partial}{\partial t} u_{\lambda\theta} + \frac{1}{2}(P_0^{\alpha}\Pi^{\beta} - P_0^{\beta}\Pi^{\alpha})\Big]$$

$$\rho_0 (T - T_0) = \eta^{\alpha\beta} u_{\alpha\beta} + \nu^{\alpha\beta}\partial_{\alpha} p_{\beta} + \nu s_1 + \beta^{\alpha} p_{\alpha}$$

$$\rho_0 T_0 \frac{\partial s_1}{\partial t} = \varkappa^{\alpha\beta}\partial_{\alpha}\partial_{\beta} T, \quad \frac{\partial}{\partial t}\rho_1 + \rho_0 \partial_{\alpha} v^{\alpha} = 0$$

**3. Isentropic waves in elastic polarizing media.** In an electrostatic approximation let us consider the class of models of elastic polarizing media for which the function $\Lambda_0$ has the form

$$-\Lambda_0 = \frac{2\pi}{\varepsilon - 1} P_{\alpha} P^{\alpha} + f(P_{\alpha}^{\wedge}, K_C) + \lambda_0^{\alpha\beta\lambda\theta}\varepsilon_{\alpha\beta}^{\wedge}\varepsilon_{\lambda\theta}^{\wedge} + \zeta_0^{\alpha\beta\lambda}\varepsilon_{\alpha\beta}^{\wedge} P_{\lambda}^{\wedge} + \lambda_0^{\alpha\beta}\varepsilon_{\alpha\beta}^{\wedge} \tag{3.1}$$

where $P_{\alpha}^{\wedge}$ are components of the polarization vector of the medium evaluated in the appropriate coordinate system, $f$ is a given function of the arguments noted in (3.1) that governs the anisotropy energy, $\varepsilon$ is a given constant; the constant components of the tensors $K_C$, $\lambda_0^{\alpha\beta\lambda\theta}$, $\lambda_0^{\alpha\beta}$, $\zeta_0^{\alpha\beta\lambda}$ are given as a function of the form of symmetry of the medium. In linearized theory, the function $-\Lambda_0$ defined by (3.1) is written in the form of (2.2) in which

$$\lambda^{\alpha\beta\lambda\theta} = \lambda_0^{\alpha\beta\lambda\theta} + \zeta^{\alpha\beta\theta} P_0^{\lambda} + \zeta^{\lambda\theta\beta} P_0^{\alpha} - \frac{1}{2}\delta^{\alpha\lambda}\left[P_0^{\theta}\left(\frac{\partial f}{\partial P_{\beta}^{\wedge}}\right)_0 +\right. \tag{3.2}$$
$$\left. P_0^{\beta}\left(\frac{\partial f}{\partial P_{\theta}^{\wedge}}\right)_0\right] + P_0^{\alpha} P_0^{\lambda}\left(\frac{\partial f}{\partial P_{\beta}^{\wedge}\partial P_{\theta}^{\wedge}}\right)_0 + \frac{1}{2}\delta^{\alpha\lambda}(Q^{\theta\beta} + Q^{\beta\theta}) + \delta^{\alpha\theta} Q^{\lambda\beta} + \delta^{\lambda\beta} Q^{\alpha\theta}$$

$$\zeta^{\alpha\beta\lambda} = \zeta_0^{\alpha\beta\lambda} + \delta^{\alpha\lambda}\left(\frac{\partial f}{\partial P_{\beta}^{\wedge}}\right)_0 + P_0^{\alpha}\left(\frac{\partial f}{\partial P_{\beta}^{\wedge}\partial P_{\lambda}^{\wedge}}\right)_0$$

$$C^{\alpha} = \frac{4\pi}{\varepsilon - 1} P_0^{\alpha} + \left(\frac{\partial f}{\partial P_{\alpha}^{\wedge}}\right)_0, \quad \beta^{\alpha\beta} = \frac{4\pi}{\varepsilon - 1}\delta^{\alpha\beta} + \left(\frac{\partial f}{\partial P_{\alpha}^{\wedge}\partial P_{\beta}^{\wedge}}\right)_0$$

$$Q^{\alpha\beta} = \zeta_0^{\alpha\beta\lambda} P_{0\lambda} + P_0^{\alpha}\left(\frac{\partial f}{\partial P_{\beta}^{\wedge}}\right)_0 + \lambda_0^{\alpha\beta}, \qquad \alpha^{\alpha\beta\lambda\theta} = \mu^{\alpha\beta\lambda\theta} = Q^{\alpha\beta\lambda} = C^{\alpha\beta} = 0$$

The parentheses $(\ )_0$ here denotes that the function in the parentheses is evaluated in the initial state. Furthermore, we examine the case (corresponding to piezoceramic media, for instance) when the polarizing medium possesses axial symmetry, there are no external forces $(Q_{\alpha} = 0)$, and the function $f$ and coefficients $\zeta_0^{\alpha\beta\lambda}$, $\lambda_0^{\alpha\beta}$, $\lambda_0^{\alpha\beta\lambda\theta}$ in $\Lambda_0$ and the coefficients $S^{\alpha\beta}$, $\tau^{\alpha\beta\lambda\theta}$ in (2.3) are defined by the relationships

$$f = \frac{1}{2}(\beta_1^{\circ}\delta^{\alpha\beta} + \beta_2^{\circ} n^{\alpha} n^{\beta}) P_{\alpha}^{\wedge} P_{\beta}^{\wedge}, \quad s^{\alpha\beta} = \tau_{\perp}(\delta^{\alpha\beta} - n^{\alpha} n^{\beta}) + \tau_{\parallel} n^{\alpha} n^{\beta} \tag{3.3}$$

$$\zeta_0^{\alpha\beta\lambda} = \zeta_1^{\circ} n^{\alpha} n^{\beta} n^{\lambda} + \zeta_2^{\circ}\delta^{\alpha\beta} n^{\lambda} + \zeta_3^{\circ}(\delta^{\alpha\lambda} n^{\beta} + \delta^{\beta\lambda} n^{\alpha}), \quad \tau^{\alpha\beta\lambda\theta} = 0$$

$$\lambda_0^{\alpha\beta\lambda\theta} = \lambda_1^{\circ}\delta^{\alpha\beta}\delta^{\lambda\theta} + \lambda_2^{\circ}(\delta^{\alpha\lambda}\delta^{\beta\theta} + \delta^{\alpha\theta}\delta^{\beta\lambda}) + \lambda_3^{\circ}(\delta^{\alpha\lambda} n^{\beta} n^{\theta} +$$
$$\delta^{\alpha\theta} n^{\beta} n^{\lambda} + \delta^{\lambda\beta} n^{\alpha} n^{\theta} + \delta^{\beta\theta} n^{\alpha} n^{\lambda}) + \lambda_4^{\circ}(\delta^{\alpha\beta} n^{\lambda} n^{\theta} + \delta^{\lambda\theta} n^{\alpha} n^{\beta}) +$$
$$\lambda_5^{\circ} n^{\alpha} n^{\beta} n^{\lambda} n^{\theta}, \quad \lambda_0^{\alpha\beta} = a_1\delta^{\alpha\beta} + a_2 n^{\alpha} n^{\beta}$$

in which $n^{\alpha}$ are components of the unit vector directed along the anisotropy axis, and $\beta^{\circ}$, $\tau$, $\zeta^{\circ}$, $\lambda^{\circ}$, $a$ are constants. Assuming the vector of constant polarization of the medium directed along the anisotropy axis $P_0^{\alpha} = P_0 n^{\alpha}$, we find for the coefficients $\zeta^{\alpha\beta\lambda}$, $\beta^{\alpha\beta}$, $\lambda^{\alpha\beta\lambda\theta}$, $C^{\alpha}$, $Q^{\alpha\beta}$

$$\beta^{\alpha\beta} = \beta_1\delta^{\alpha\beta} + \beta_2 n^{\alpha} n^{\beta}, \ \zeta^{\alpha\beta\lambda} = \zeta_1 n^{\alpha} n^{\beta} n^{\lambda} + \zeta_2\delta^{\alpha\beta} n^{\lambda} + \zeta_3\delta^{\beta\lambda} n^{\alpha} + \zeta_4\delta^{\alpha\lambda} n^{\beta} \tag{3.4}$$

$$Q^{\alpha\beta} = [a_2 + P_0(\zeta_1^{\circ} + 2\zeta_3^{\circ}) + P_0^2(\beta_1^{\circ} + \beta_2^{\circ})]\, n^{\alpha} n^{\beta} + (a_1 + P_0\zeta_2^{\circ})\,\delta^{\alpha\beta}, \quad C^{\alpha} = \beta^{\alpha\beta} P_{0\beta}$$

$$\lambda^{\alpha\beta\lambda\theta} = \lambda_1\delta^{\alpha\beta}\delta^{\lambda\theta} + \lambda_2(\delta^{\alpha\lambda}\delta^{\beta\theta} + \delta^{\alpha\theta}\delta^{\beta\lambda}) + \lambda_3\delta^{\alpha\lambda} n^{\beta} n^{\theta} +$$
$$\lambda_4(\delta^{\alpha\theta} n^{\beta} n^{\lambda} + \delta^{\lambda\beta} n^{\alpha} n^{\theta}) + \lambda_5\delta^{\beta\theta} n^{\alpha} n^{\lambda} + \lambda_6(\delta^{\alpha\beta} n^{\lambda} n^{\theta} + \delta^{\lambda\theta} n^{\alpha} n^{\beta}) + \lambda_7 n^{\alpha} n^{\beta} n^{\lambda} n^{\theta}$$

The coefficients $\lambda$, $\zeta$, $\beta$ in (3.4) are related to the coefficients $\lambda^\circ$, $\zeta^\circ$, $\beta^\circ$ in (3.3) by the equalities

$$\beta_1=\beta_1^\circ+\frac{4\pi}{\varepsilon-1},\quad \beta_2=\beta_2^\circ,\ \zeta_1=\zeta_1^\circ+P_0\beta_2^\circ,\quad \zeta_2=\zeta_2^\circ,\quad \zeta_3=\zeta_3^\circ+P_0\beta_1^\circ \tag{3.5}$$
$$\zeta_4=\zeta_3^\circ+(\beta_1^\circ+\beta_2^\circ)P_0,\ \lambda_1=\lambda_1^\circ,\ \lambda_2=\lambda_2^\circ,\ \lambda_3=\lambda_3^\circ-P_0^2(\beta_1^\circ+\beta_2^\circ)$$
$$\lambda_4=\lambda_3^\circ+P_0\zeta_3^\circ,\quad \lambda_5=\lambda_3^\circ+P_0(2\zeta_3^\circ+P_0\beta_1^\circ)\ \lambda_6=\lambda_4^\circ+P_0\zeta_2^\circ,\quad \lambda_7=\lambda_5^\circ+P_0(2\zeta_1^\circ+P_0\beta_2^\circ)$$

The condition of no stresses in the initial state ($Q^{\alpha\beta}=0$) is satisfied if the coefficients $a_1$, $a_2$ in (3.3) are defined by the equations

$$a_1+P_0\zeta_2^\circ=0,\quad a_2+P_0(\zeta_1^\circ+2\zeta_3^\circ)+P_0^2(\beta_1^\circ+\beta_2^\circ)=0 \tag{3.6}$$

Let us note that the components of the tensor $\xi^{\alpha\beta\lambda}$ in $\Lambda_0$ governing the piezoelectrical energy are not symmetric in the superscripts $\alpha,\beta$ in the general case, hence, the equation for the polarization in (2.4) relates the components of the electrical field intensity vector $e^\alpha$ to not only the strain tensor components $\varepsilon_{\alpha\beta}={}^1/_2(u_{\alpha\beta}+u_{\beta\alpha})$ (as in ordinary linear theories) but also the components of the rotation vector of the strain axes $\Omega_\alpha={}^1/_2\,\mathrm{rot}_\alpha u$. In the same way, the elastic energy ${}^1/_2\lambda^{\alpha\beta\lambda\theta}u_{\alpha\beta}u_{\lambda\theta}$ depends on both $\varepsilon_{\alpha\beta}$ and on $\Omega_\alpha$.

Assuming the relationships (2.3), (3.1)—(3.7) satisfied, let us examine the solution of equations (2.4) in the form of plane isentropic waves

$$p^\alpha=p_0^\alpha\exp i(k_\lambda x^\lambda-\omega t),\quad u^\alpha=u_0^\alpha\exp i(k_\lambda x^\lambda-\omega t),\quad e^\alpha=e_0^\alpha\exp i(k_\lambda x^\lambda-\omega t)$$

were $k_\lambda$ are the wave vector components, $\omega$ is the wave frequency, and $p_0^\alpha$, $u_0^\alpha$, $e_0^\alpha$ are constant amplitudes. We first examine the case when $k_\lambda=kn_\lambda$. From the Maxwell equations, the equations for the polarization, and from the momentum equations in (2.4) we find

$$e^\alpha=-4\pi k^{-2}k^\alpha k_\lambda p^\lambda,\quad p^\alpha=\eta_1e^\alpha+\eta_2n^\alpha n_\lambda e^\lambda+\theta_1u^\alpha+\theta_2n^\alpha n_\lambda u^\lambda \tag{3.7}$$
$$\Big\{\Big[\rho_0\omega^2-k^2\Big(\lambda_2+\lambda_3-\frac{i}{4}P_0^2\omega\tau_\perp\Big)\Big]\delta^{\alpha\beta}-$$
$$(\lambda_1+\lambda_2+2\lambda_4+\lambda_5+2\lambda_6+\lambda_7+\frac{i}{4}P_0^2\omega\tau_\perp\Big)k^2n^\alpha n^\beta\Big\}u_\beta+$$
$$k\Big\{\Big(i\zeta_4-\frac{1}{2}P_0\omega\tau_\perp\Big)\delta^{\alpha\beta}+\Big[i(\zeta_1+\zeta_2+\zeta_3)+\frac{1}{2}P_0\omega\tau_\perp\Big]n^\alpha n^\beta\Big\}p_\beta=0$$

The coefficients $\eta$, $\theta$ are defined as follows:

$$\eta_1=\frac{1}{\beta_1-i\omega\tau_\perp},\quad \eta_2=-\eta_1+\frac{1}{\beta_1+\beta_2-i\omega\tau_\parallel} \tag{3.8}$$
$$\theta_1=k\eta_1\Big(-i\xi_4+\frac{1}{2}P_0\omega\tau_\perp\Big),\ \theta_2=-\theta_1-ik(\eta_1+\eta_2)(\zeta_1+\zeta_2+\zeta_3+\zeta_4)$$

A dispersion equation for the longitudinal (3.9) and transverse (3.10) waves follows from (3.7)

$$\rho_0\omega^2=k^2\Big[\lambda_1+2\lambda_2+\lambda_3+2\lambda_4+\lambda_5+2\lambda_6+\lambda_7-\frac{(\zeta_1+\zeta_2+\zeta_3+\zeta_4)^2}{4\pi+\beta_1+\beta_2-i\omega\tau_\parallel}\Big] \tag{3.9}$$

$$\rho_0\omega^2=k^2\Big[\lambda_2+\lambda_3+P_0\zeta_4+\frac{1}{4}P_0^2\beta_1-\frac{(\zeta_4+{}^1/_2P_0\beta_1)^2}{\beta_1-i\omega\tau_\perp}\Big] \tag{3.10}$$

If the axis $x^3$ of the coordinate system is directed along the constant polarization vector of the medium $P_0$, then we have for the longitudinal wave described by the dispersion equation (3.9)

$$u^\alpha=(0,0,u),\quad p^\alpha=(0,0,p),\quad e^\alpha=(0,0,4\pi p) \tag{3.11}$$

The quantities $u$, $p$ in (3.11) are connected by the relationship

$$p=-u\,\frac{ik(\zeta_1+\zeta_2+\zeta_3+\zeta_4)}{4\pi+\beta_1+\beta_2-i\omega\tau_\parallel} \tag{3.12}$$

For the transverse wave described by (3.10), we have

$$\rho_1=0,\quad e^\alpha=0,\quad u^\alpha=(u^1,u^2,0),\quad p^\alpha=(p^1,p^2,0) \tag{3.13}$$

where

$$p^\alpha=u^\alpha\,\frac{k}{2}\,\frac{P_0\omega\tau_\perp-2i\zeta_4}{\beta_1-i\omega\tau_\perp} \tag{3.14}$$

In the case when the wave vector $k$ is orthogonal to the constant polarization vector of the medium $k_\alpha n^\alpha = 0$, the dispersion equation for the longitudinal wave has the form

$$\rho_0\omega^2 = k^2\left(\lambda_1 + 2\lambda_2 - \frac{\zeta_2^2}{\beta_1+\beta_2-i\omega\tau_\parallel}\right) \qquad (3.15)$$

In the case when $k_\alpha n^\alpha = 0$, the dispersion equations for the transverse waves are written as follows

$$\rho_0\omega^2 = \lambda_2 k^2 \qquad (3.16)$$

$$\rho_0\omega^2 = k^2\left\{\lambda_2 + \lambda_5 - P_0\zeta_3 + \frac{1}{4}P_0^2(4\pi+\beta_1) - \frac{[\zeta_3 - {}^1/_2 P_0(4\pi+\beta_1)]^2}{4\pi+\beta_1-i\omega\tau_\perp}\right\}$$

The relation between $p^\alpha$, $u^\alpha$ in the case under consideration has the form

$$p^\alpha = -\frac{1}{2}\frac{2i\zeta_3 + P_0\omega\tau_\perp}{4\pi+\beta_1-i\omega\tau_\perp} n_\beta u^\beta k^\alpha - \frac{i\zeta_2}{\beta_1+\beta_2-i\omega\tau_\parallel} n^\alpha k_\lambda u^\lambda \qquad (3.17)$$

The dispersion equations (3.9), (3.10), (3.15), (3.16) are third order equations in the frequency $\omega$ and of second order in the wave vector components $k_\lambda$ in the general case (solved explicitly for $k^2$). It is seen from (3.9), (3.10), (3.15) and (3.16) that the coefficient $\tau_\perp$ in the relaxation term $\Pi^\alpha$ of the equation for the electrical polarization of the medium governs the transverse wave attenuation, while the coefficient $\tau_\parallel$ governs longitudinal wave attenuation. If relaxation of the electrical polarization is not taken into account $\Pi^\alpha = 0$ ($\tau_\perp = \tau_\parallel = 0$), then the dispersion equations (3.9), (3.10), (3.15), (3.16) define the customary elastic waves whose propagation velocity depends on the constant coefficients $\zeta$, $\beta$ in $\Lambda_0$, governing the anisotropy energy and the piezoelectrical energy. Let us note that for $\zeta_i = 0$ (when the piezoelectrical energy is not taken into account), attenuation of the longitudinal waves considered also does not occur; attenuation of the transverse waves occurs even in the absence of the piezoelectric effect.

All the dispersion equations obtained above have the form

$$\rho_0\omega^2 = k^2\left(a - \frac{b}{c-i\omega\tau}\right) \qquad (3.18)$$

where $a$, $b$, $c$, $\tau$ are certain positive constant. In application to elastic polarizing media such dispersion equations were considered (for $\tau = 0$) in /1/, for instance. The equation (3.18) yields the complex value $\omega = \omega_0 - i\gamma$ for $\omega$ in taking account of polarization relaxation when $\tau \neq 0$, where the decrement $\gamma$ governs the wave attenuation. From (3.18) we obtain that the decrement $\gamma$ is related to the wave vector as follows:

$$(\tau k)^2 = -2\rho_0\tau\gamma\frac{(2\tau\gamma - c)^2}{2a\tau\gamma - b} \qquad (3.19)$$

A graph of the function $\tau\gamma = f(\tau k)$ defined according to (3.19) in the physically real case when $b/a < c$ is a monotonically growing curve tangent to the axis $\tau k$ at the point 0 and having the horizontal asymptote $\tau\gamma = b/2a$.

If the spatial inhomogeneity of the electrical polarization of the medium is taken into account (by inserting the term $\alpha^{\alpha\lambda}\hat{\nabla}_\alpha \hat{P}_\beta \cdot \hat{\nabla}_\lambda P^{\wedge\beta}$ in (3.1) for $\Lambda_0$), then the dispersion equations (3.9), (3.10), (3.15), (3.16) retain their form if the coefficient $\beta_1$ therein is replaced by the coefficient $\beta_1^*$ defined by $\beta_1^* = \beta_1 + \alpha^{\alpha\lambda}k_\alpha k_\lambda$.

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